The Series Rearrangement Theorem

A Rigorous Proof and Visualization

Abstract

We formalize a rigorous proof of the Series Rearrangement Theorem based on partitioning a conditionally convergent series into its non-negative and strictly negative subsequences. We inductively construct a sequence of partial sums that alternates leaping across a target value α, explicitly demonstrating that the oscillation error is bounded by the sequence's final jump sizes.

1 Proof of the Theorem

Theorem 1 (Rearrangement Theorem)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *be a conditionally convergent series of real numbers. For any real number* α*, there exists a bijection* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that*

$$\sum_{m=1}^{\infty} a_{\sigma(m)} = \alpha.$$

Proof. **Step 1: Separating the positive and negative terms.**
Let $(p_n)_{n=1}^{\infty}$ be the subsequence of all non-negative terms of (a_n), and let $(q_n)_{n=1}^{\infty}$ be the subsequence consisting of the absolute values of all strictly negative terms of (a_n). By definition, $p_n \geq 0$ and $q_n > 0$, and every term of the original sequence (a_n) belongs to exactly one of these two subsequences.

We are given that $\sum a_n$ converges to a finite value L, and $\sum |a_n|$ diverges to ∞. Let P_N and Q_N be the sums of all non-negative and negative components among the first N terms of a_n, respectively. We can write:

$$\sum_{n=1}^{N} a_n = P_N - Q_N \quad \text{and} \quad \sum_{n=1}^{N} |a_n| = P_N + Q_N.$$

Since $P_N - Q_N \to L$ and $P_N + Q_N \to \infty$, adding and subtracting the sequences reveals that both $P_N \to \infty$ and $Q_N \to \infty$. Thus, the infinite series of the separated subsequences both diverge: $\sum_{n=1}^{\infty} p_n = \infty$ and $\sum_{n=1}^{\infty} q_n = \infty$.

Furthermore, the convergence of the original series implies that its general term vanishes ($\lim_{n\to\infty} a_n = 0$). Since they are subsequences, we must also have $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} q_n = 0$.

Step 2: Inductive construction of the rearrangement.
Without loss of generality, we assume $\alpha \geq 0$. (If $\alpha < 0$, we could easily start the sequence by subtracting terms of q_n instead, or simply apply the following proof to the sequence $(-a_n)$ targeting $-\alpha$).

We construct two strictly increasing sequences of indices $(k_j)_{j=0}^{\infty}$ and $(s_j)_{j=0}^{\infty}$ with $k_0 = 0$ and $s_0 = 0$. Let the initial partial sum be $y_0 = 0$. Note that $y_0 \leq \alpha$.

For each step $j \geq 1$:
Because $\sum p_n = \infty$, we define k_j as the *smallest* integer strictly greater than k_{j-1} such that adding terms forces the sum strictly past α:

$$x_j := y_{j-1} + \sum_{i=k_{j-1}+1}^{k_j} p_i > \alpha.$$

Because k_j is the minimal index required to exceed α, the partial sum immediately prior to adding p_{k_j} must have been less than or equal to α. (For $j = 1$, the prior sum is bounded by our assumption that $y_0 = 0 \leq \alpha$). Thus, $x_j - p_{k_j} \leq \alpha$. Rearranging this, and noting $x_j > \alpha$, yields:

$$0 < x_j - \alpha \leq p_{k_j}. \tag{1}$$

Next, since $\sum q_n = \infty$, we define s_j as the *smallest* integer strictly greater than s_{j-1} such that subtracting terms drops the sum strictly below α:

$$y_j := x_j - \sum_{i=s_{j-1}+1}^{s_j} q_i < \alpha.$$

By the minimality of s_j, the sum prior to subtracting q_{s_j} must have been $\geq \alpha$. Thus, $y_j + q_{s_j} \geq \alpha$. Since $y_j < \alpha$, this gives:

$$0 < \alpha - y_j \leq q_{s_j}. \tag{2}$$

Step 3: Convergence of the rearranged series.
We construct our rearranged sequence $(a_{\sigma(m)})_{m=1}^{\infty}$ by alternating the blocks we defined:

$$p_1, \ldots, p_{k_1}, -q_1, \ldots, -q_{s_1}, p_{k_1+1}, \ldots, p_{k_2}, -q_{s_1+1}, \ldots$$

Because (k_j) and (s_j) strictly increase indefinitely, every term of (p_n) and (q_n) is eventually included exactly once. Thus, $\sigma : \mathbb{N} \to \mathbb{N}$ is a valid bijection of the original sequence.

Let S_N be the N-th partial sum of this rearranged series. For a sufficiently large N, S_N falls inside either the j-th block of positive terms or the j-th block of negative terms.

- **Inside a positive block ($j \geq 1$):** The sequence of partial sums monotonically increases from y_{j-1} up to x_j. For $j \geq 2$, $y_{j-1} < \alpha < x_j$, so the distance from S_N to α is bounded by the distance to the furthest boundary:

  $$|S_N - \alpha| \leq \max(\alpha - y_{j-1}, x_j - \alpha) \leq \max(q_{s_{j-1}}, p_{k_j}).$$

  (For $j = 1$, since $S_N \in [0, x_1]$, $|S_N - \alpha| \leq \max(\alpha, p_{k_1})$).

- **Inside a negative block ($j \geq 1$):** The sequence of partial sums monotonically decreases from x_j down to y_j. Since $y_j < \alpha < x_j$:

  $$|S_N - \alpha| \leq \max(x_j - \alpha, \alpha - y_j) \leq \max(p_{k_j}, q_{s_j}).$$

As $N \to \infty$, the block index $j \to \infty$. Because $\lim p_n = 0$ and $\lim q_n = 0$, their active jumping terms must also vanish: $\lim_{j\to\infty} p_{k_j} = 0$ and $\lim_{j\to\infty} q_{s_j} = 0$. Consequently, the upper bounds on $|S_N - \alpha|$ converge strictly to 0. This rigorously proves that $\lim_{N\to\infty} S_N = \alpha$. □

2 Geometric Visualization of the Inductive Jumps

The mechanism of the proof relies on crossing α and immediately stopping. Because we stop the very first time we cross α, our overshoot is necessarily bounded by the magnitude of the single term that pushed us across.

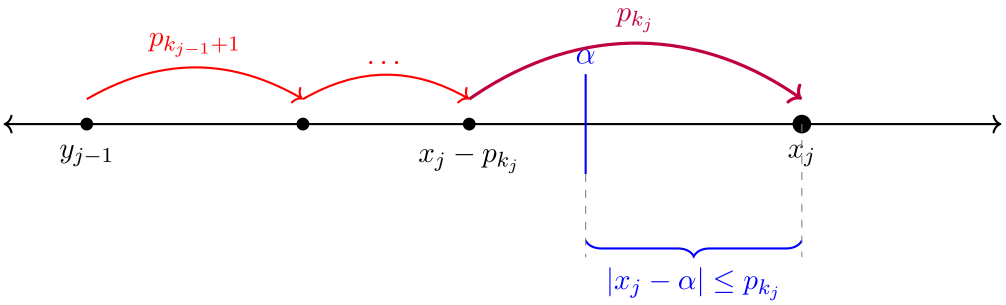

Figure 1: Adding non-negative terms strictly increases the partial sum until it crosses α. The term p_{k_j} is the *first* term to push the sum past α. Therefore, the previous sum $x_j - p_{k_j}$ was $\leq \alpha$, which geometrically forces the overshoot distance $|x_j - \alpha|$ to be strictly smaller than the final jump p_{k_j}.

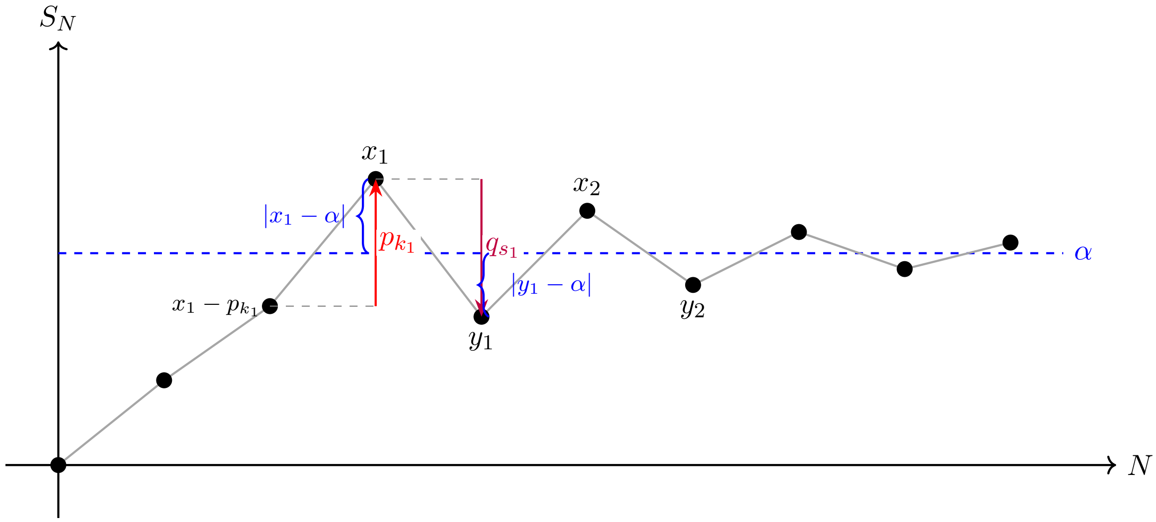

Figure 2: The sequence of partial sums S_N oscillating around α. Because the sequence turns back immediately when it crosses the threshold, the maximum distance from α at the turning points x_j and y_j is firmly bounded by the final jump. Since $\lim p_{k_j} = 0$ and $\lim q_{s_j} = 0$, the oscillation envelope eventually squeezes to 0.