Products Preserve Riemann Integrability

A Local Oscillation Proof

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A proof about boundedness, local variation, and how Darboux estimates are built from pointwise identities.

1. Why prove the product theorem this way?

The theorem is familiar: if f and g are Riemann integrable on a compact interval, then their product fg is Riemann integrable. There are several standard ways to prove this. One can use upper and lower step-function approximations, one can split functions into positive and negative parts, or one can appeal to a more general theorem saying that continuous functions of Riemann integrable functions are Riemann integrable.

Those approaches are valuable, but they can hide the most useful local mechanism. The proof below takes a different point of view. It asks the concrete question:

If x and y lie in the same small interval, how much can $f(x)g(x)$ differ from $f(y)g(y)$?

The answer is encoded in the elementary identity

$$f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)).$$

This identity is the engine of the proof. It separates the motion of the product into two parts: the motion of f, multiplied by a bounded value of g, and the motion of g, multiplied by a bounded value of f.

Thus, on every interval I, the local oscillation of the product is controlled by the local oscillations of the factors:

$$\mathrm{osc}_I(fg) \leq M_g \, \mathrm{osc}_I(f) + M_f \, \mathrm{osc}_I(g),$$

where M_f and M_g are global bounds for $|f|$ and $|g|$.

Once this local estimate is known, the Darboux proof becomes almost automatic: choose a common refinement on which the oscillation sums for f and g are small, and the oscillation sum for fg will be small too.

The point of the note is therefore not just to prove a closure property. The point is to practice a transferable proof pattern:

expand a difference $\longrightarrow$ isolate the moving pieces $\longrightarrow$ use boundedness $\longrightarrow$ sum local estimates globally.

This is a flexible analytic maneuver. It avoids sign-splitting and does not rely on multiplication being order-preserving for arbitrary upper and lower approximants. It replaces an order problem with a variation problem.

2. Darboux notation and oscillation

Let $u : [a, b] \to \mathbb{R}$ be bounded. If $I \subseteq [a, b]$ is a subinterval, define the oscillation of u on I by

$$\operatorname{osc}_I(u) = \sup_{t \in I} u(t) - \inf_{t \in I} u(t).$$

Let

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b,$$

be a partition of $[a, b]$. Write

$$I_i = [x_{i-1}, x_i], \qquad \Delta x_i = x_i - x_{i-1}.$$

Then

$$U(P, u) - L(P, u) = \sum_{i=1}^{n} \operatorname{osc}_{I_i}(u) \Delta x_i.$$

We shall use the Darboux criterion: a bounded function u is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, u) - L(P, u) < \varepsilon.$$

The following lemma is a convenient way to estimate oscillations. It lets us replace upper and lower extrema by arbitrary pairs of points.

Lemma 1 (Oscillation as a two-point supremum)**.** *Let $u : I \to \mathbb{R}$ be bounded on an interval I. Then*

$$\operatorname{osc}_I(u) = \sup_{x,y \in I} |u(x) - u(y)|.$$

Proof. Let

$$M = \sup_{t \in I} u(t), \qquad m = \inf_{t \in I} u(t).$$

For every $x, y \in I$,

$$|u(x) - u(y)| \leq M - m,$$

and therefore

$$\sup_{x,y \in I} |u(x) - u(y)| \leq M - m.$$

Conversely, fix $\eta > 0$. By the defining properties of the supremum and infimum, there exist $x, y \in I$ such that

$$u(x) > M - \frac{\eta}{2}, \qquad u(y) < m + \frac{\eta}{2}.$$

Hence

$$|u(x) - u(y)| \geq u(x) - u(y) > M - m - \eta.$$

Since $\eta > 0$ was arbitrary,

$$\sup_{x,y \in I} |u(x) - u(y)| \geq M - m.$$

Combining the two inequalities gives

$$\operatorname{osc}_I(u) = \sup_{x,y \in I} |u(x) - u(y)|.$$

□

3. The local product estimate

The central estimate is purely local. It does not use integrability. It only uses boundedness and the product-difference identity.

Lemma 2 (Product oscillation estimate)**.** *Let $f, g : I \to \mathbb{R}$ be bounded functions on an interval I. Suppose that*

$$|f(t)| \le M_f, \qquad |g(t)| \le M_g$$

for every $t \in I$. Then

$$\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g).$$

Proof. Let $x, y \in I$. We write

$$\begin{aligned} f(x)g(x) - f(y)g(y) &= f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y) \\ &= (f(x) - f(y))g(x) + f(y)(g(x) - g(y)). \end{aligned}$$

Taking absolute values, and using the bounds on f and g, gives

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x) - f(y)|\,|g(x)| + |f(y)|\,|g(x) - g(y)| \\ &\le M_g|f(x) - f(y)| + M_f|g(x) - g(y)|. \end{aligned}$$

Taking the supremum over all $x, y \in I$ and using the preceding lemma, we obtain

$$\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g).$$

□

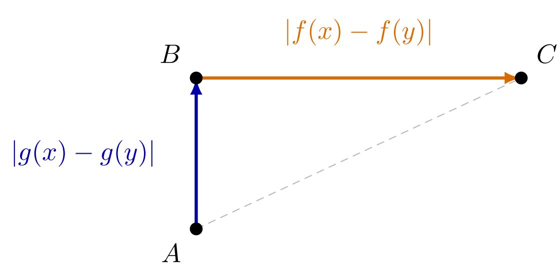


Figure 1: A schematic view of the product-difference identity. Here $A = (f(y), g(y))$, $B = (f(y), g(x))$, and $C = (f(x), g(x))$. Moving from A to C through B first changes the g-coordinate and then changes the f-coordinate. The vertical move is weighted by $|f(y)| \le M_f$, and the horizontal move is weighted by $|g(x)| \le M_g$.

4. Products preserve Riemann integrability

Theorem 1. *Let $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions. Then fg is Riemann integrable on $[a, b]$.*

Proof. Since Riemann integrable functions on compact intervals are bounded, choose constants $M_f, M_g \geq 0$ such that

$$|f(x)| \leq M_f, \qquad |g(x)| \leq M_g$$

for every $x \in [a, b]$.

Fix $\varepsilon > 0$. Choose

$$\rho = \frac{\varepsilon}{M_f + M_g + 1}.$$

Since f and g are Riemann integrable, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \rho$$

and

$$U(P_2, g) - L(P_2, g) < \rho.$$

Let

$$P = P_1 \cup P_2 = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b.$$

Because refinement decreases the difference between the upper and lower Darboux sums, we have

$$U(P, f) - L(P, f) < \rho, \qquad U(P, g) - L(P, g) < \rho.$$

For each i, write

$$I_i = [x_{i-1}, x_i], \qquad \Delta x_i = x_i - x_{i-1}.$$

By the product oscillation estimate,

$$\mathrm{osc}_{I_i}(fg) \leq M_g \ \mathrm{osc}_{I_i}(f) + M_f \ \mathrm{osc}_{I_i}(g).$$

Multiplying by Δx_i and summing gives

$$\begin{aligned}
U(P, fg) - L(P, fg) &= \sum_{i=1}^{n} \mathrm{osc}_{I_i}(fg)\Delta x_i \\
&\leq M_g \sum_{i=1}^{n} \mathrm{osc}_{I_i}(f)\Delta x_i + M_f \sum_{i=1}^{n} \mathrm{osc}_{I_i}(g)\Delta x_i \\
&= M_g\big(U(P, f) - L(P, f)\big) + M_f\big(U(P, g) - L(P, g)\big) \\
&< M_g\rho + M_f\rho \\
&= (M_f + M_g)\rho \\
&< \varepsilon.
\end{aligned}$$

Therefore, for every $\varepsilon > 0$, there exists a partition P such that

$$U(P, fg) - L(P, fg) < \varepsilon.$$

By the Darboux criterion, fg is Riemann integrable on $[a, b]$. □

5. What the proof is really doing

The entire proof is organized around the estimate

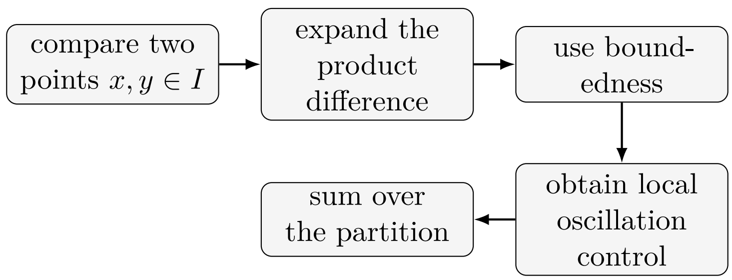
$$\boxed{\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g)}$$

for every subinterval $I \subseteq [a, b]$.

This estimate converts local information about f and g into local information about fg. Once we multiply by interval lengths and sum, the local estimate becomes a Darboux estimate:

$$U(P, fg) - L(P, fg) \le M_g\big(U(P, f) - L(P, f)\big) + M_f\big(U(P, g) - L(P, g)\big).$$

So the argument has a clean local-to-global structure:



This is a useful proof pattern because it works before one has found a polished global abstraction. Instead of trying to guess the right theorem in advance, we ask what the expression does between two nearby points. The local algebra reveals the estimate, and the Darboux machinery turns that estimate into integrability.

6. Why this avoids sign-splitting

There is a subtle reason some upper/lower approximation proofs of the product theorem become longer than one might expect. If

$$\underline{f} \le f \le \overline{f}, \qquad \underline{g} \le g \le \overline{g},$$

it is not generally true that

$$\underline{f}\,\underline{g} \le fg \le \overline{f}\,\overline{g}.$$

Multiplication is monotone in each variable only under sign restrictions. This is why one often splits into positive and negative parts before multiplying minorants and majorants.

The oscillation proof bypasses that issue. It does not ask whether products of minorants minorize or products of majorants majorize. It asks instead:

How far apart can two values of the product be on the same small interval?

This question is answered directly by

$$f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)).$$

No positivity is required. The possible sign changes of f and g are absorbed by absolute values and boundedness.

Thus the proof replaces an order problem with a variation problem. That is why the method is robust.

7. The reusable template

The product theorem is one instance of a broader local principle. Suppose $F : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz on a bounded region containing the range of

$$(f_1(x), \dots, f_m(x)).$$

Then for x, y in the same interval,

$$\begin{aligned} &|F(f_1(x), \dots, f_m(x)) - F(f_1(y), \dots, f_m(y))| \\ &\qquad \le C\left(|f_1(x) - f_1(y)| + \cdots + |f_m(x) - f_m(y)|\right) \end{aligned}$$

for some constant C. Taking suprema over x, y in the same interval gives a local oscillation estimate. Summing over a partition gives a Darboux estimate.

For multiplication, $F(s,t) = st$ is not globally Lipschitz on all of $\mathbb{R}^2$, but it is Lipschitz on every bounded rectangle. Since Riemann integrable functions on compact intervals are bounded, this is enough.

For maximum, $F(s,t) = \max(s,t)$ is globally Lipschitz, and the same philosophy gives

$$\operatorname{osc}_I(\max(f,g)) \le \operatorname{osc}_I(f) + \operatorname{osc}_I(g).$$

So the product theorem and the maximum theorem are not isolated tricks. They are examples of a single habit: control the local oscillation of a constructed function by the local oscillations of the original functions.

8. Final summary

The proof rests on one inequality:

$$\operatorname{osc}_I(fg) \le M_g \operatorname{osc}_I(f) + M_f \operatorname{osc}_I(g).$$

This inequality follows from the product-difference identity and boundedness. Once it is established, the Darboux criterion finishes the argument immediately.

The broader lesson is that Riemann integrability is not only a matter of formal closure properties. It is also a matter of controlling oscillation. When a new function is built from old ones, a powerful first move is to compare its values at two points in the same small interval, expand the difference, identify the moving pieces, and then sum the resulting local estimates.

That is the proof mechanism exposed by the product theorem.