

Exercise Problems in Mathematics

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Exercise Problems

1. For positive integers $1 \leq k \leq n$, show that the number of nondecreasing sequences $a_1 \leq a_2 \leq \cdots \leq a_k$ where $a_1, a_2, \dots, a_k$ are chosen from $\{1, 2, \dots, n\}$ is exactly $\binom{n+k-1}{k}$.
2. For a nonzero integer a and an integer b , denote $P_{a,b} = \{an + b \mid n \text{ is an integer}\}$. Is there a finite set of integers $a_1, b_1, \dots, a_n, b_n$ with $a_i \neq a_j$ for all $i \neq j$ such that $\mathbb{Z} = \bigcup_{i=1}^n P_{a_i, b_i}$?
3. Given a finite set $S = \{1, 2, \dots, n\}$ and a positive integer k , determine the number of k -tuples $(i_1, i_2, \dots, i_k)$ such that $\{i_1, i_2, \dots, i_k\}$ is a subset of S and satisfies $i_1 < i_2 < \cdots < i_k$ with the additional condition that $i_2 \geq i_1 + 2, i_3 \geq i_2 + 2, \dots, i_k \geq i_{k-1} + 2$.
4. Determine the number of binary strings of length n , where n is a positive integer, that do not contain the substring 1011.
5. Determine the real value s such that the series

$$\sum_{n=1}^{\infty} \left[(n+s) \log \left(1 + \frac{1}{n} \right) - 1 \right]$$

converges absolutely. Additionally, analyze the rate of convergence and investigate the limit to which the series converges. Bonus points if you can show that the series equals exactly $2 \log 2 + \frac{1}{2} \log \pi - 1$ for the particular value of s .

6. Imagine you have 5000 bottles of wine, and only one is poisoned. You also have a large number of prisoners available for testing the wine (this

is set in medieval times, where prisoners had no human rights). Your goal is to identify the poisoned bottle by giving small amounts of wine to the prisoners.

Here are the conditions:

- If a prisoner consumes 5mg of the poisoned wine, they will become very sick.
- If a prisoner consumes 10mg of the poisoned wine, they will die.
- Each bottle has an essentially unlimited supply of wine.
- Also, it is the king's festivity, and the results of the poison testing take one hour to appear. Since the festivity starts in an hour and ten minutes, the poison must be identified within this timeframe, before the event begins.

Given these conditions, what is the minimal number of prisoners needed to identify the poisoned wine bottle?