Lebesgue's Criterion for Riemann Integrability

A Rigorous Topological Proof of Sufficiency

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Abstract

We present a rigorous, fully self-contained proof of the sufficiency half of Lebesgue's Criterion for Riemann Integrability. By isolating the measure-theoretic "bad" behavior of discontinuities into an open quarantine with arbitrarily small total length, and invoking a single Heine-Borel compactness argument on the remaining "good" set, we construct an optimal finite partition. We detail the resolution of the topological "endpoint trap" via a shrink-wrap argument and demonstrate how Darboux sums neatly split into two orthogonal ε-budgeting regimes.

1 The Theorem and Formal Proof

Theorem 1 (Lebesgue's Criterion - Sufficiency)**.** *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If f is continuous almost everywhere (i.e., the set of discontinuities $E \subset [a,b]$ has Lebesgue measure zero), then f is Riemann integrable on $[a,b]$.*

Proof. **Step 1: Setup and Tolerances.**
Because f is bounded, let $M = \sup_{x\in[a,b]} |f(x)|$. If $M = 0$, f is identically zero and trivially integrable, so we may assume $M > 0$. Let $\varepsilon > 0$ be given. We will construct a partition P of $[a,b]$ such that the upper and lower Darboux sums satisfy $U(f,P) - L(f,P) < \varepsilon$.

Step 2: Quarantining the Discontinuities.
Let $E \subset [a,b]$ be the set of points where f is discontinuous. By hypothesis, E has Lebesgue measure zero. There exists a countable sequence of open intervals $\{(c_j, d_j)\}_{j=1}^{\infty}$ covering E such that the sum of their lengths is stringently bounded by our "bad set" budget, $\frac{\varepsilon}{4M}$:

$$E \subset \bigcup_{j=1}^{\infty}(c_j, d_j) \quad \text{and} \quad \sum_{j=1}^{\infty}(d_j - c_j) < \frac{\varepsilon}{4M}.$$

Let $U = \bigcup_{j=1}^{\infty}(c_j, d_j)$. The open set U acts as a quarantine zone for all pathological discontinuities of f.

Step 3: Local Continuity and the "Shrink-Wrap" Trick.
Let $K = [a,b] \setminus U$. Because U is open, K is a closed subset of the compact space $[a,b]$. Thus, K is intrinsically compact. For every $x \in K$, we know $x \notin E$, meaning f is continuous at x.

Therefore, there exists a radius $\delta_x > 0$ such that for all $y \in [a,b]$ satisfying $|y - x| < 2\delta_x$, we have:

$$|f(y) - f(x)| < \frac{\varepsilon}{4(b-a)}.$$

Define the open interval $I_x = (x - \delta_x, x + \delta_x)$. Note the factor of 2. If we take any two points y, z in the **closure** $\overline{I_x} \cap [a, b] = [x - \delta_x, x + \delta_x] \cap [a, b]$, their distance to x is at most $\delta_x < 2\delta_x$. By the triangle inequality:

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{\varepsilon}{2(b-a)}.$$

Thus, the oscillation of f on the **closed** set $\overline{I_x} \cap [a, b]$ is safely and strictly bounded by $\frac{\varepsilon}{2(b-a)}$.

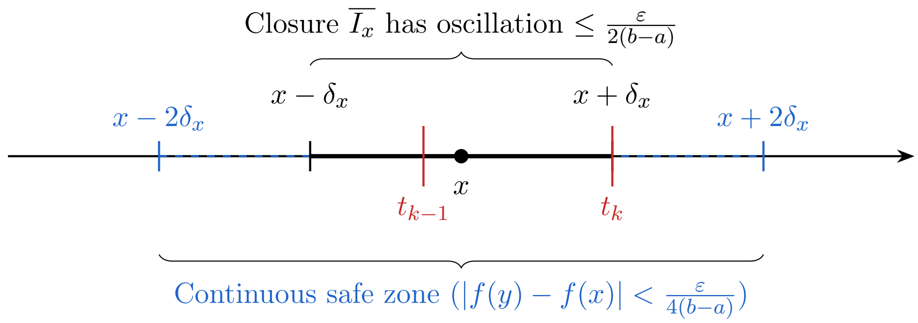

Figure 1: The **Shrink-Wrap Trick**. By extracting the open cover I_x with radius δ_x, any closed subpartition $[t_{k-1}, t_k]$ trapped inside it is heavily buffered within the $2\delta_x$ continuity window. This prevents Darboux suprema evaluations on the exact boundary from accidentally capturing an unforeseen jump discontinuity.

Step 4: The One-Shot Compactness Argument.
The collection of open intervals $\{I_x\}_{x \in K}$ covers K, while the open quarantine intervals $\{(c_j, d_j)\}_{j=1}^{\infty}$ cover $[a, b] \setminus K$. Consequently, the combined collection

$$\mathcal{C} = \{I_x\}_{x \in K} \cup \{(c_j, d_j)\}_{j=1}^{\infty}$$

is an open cover for the entire compact interval $[a, b]$. By the Heine-Borel theorem, we can extract a finite subcover:

$$\mathcal{F} = \{I_{x_1}, \ldots, I_{x_n}\} \cup \{(c_{j_1}, d_{j_1}), \ldots, (c_{j_m}, d_{j_m})\}.$$

Step 5: Inducing the Refined Partition.
Let S be the set containing a, b, and all the boundary endpoints of the intervals in $\mathcal{F}$ that fall strictly inside (a, b). Sorting the unique elements of S physically induces our partition $P = \{t_0, t_1, \ldots, t_N\}$ where $a = t_0 < t_1 < \cdots < t_N = b$.

Consider any open subinterval (t_{k-1}, t_k). Because t_{k-1} and t_k are *consecutive* elements in S, the interval (t_{k-1}, t_k) contains **no endpoints** of any interval in $\mathcal{F}$. We claim (t_{k-1}, t_k) is entirely swallowed by at least one open interval in $\mathcal{F}$.

Proof of claim: Pick $y \in (t_{k-1}, t_k)$. Since $\mathcal{F}$ covers $[a, b]$, y belongs to some interval $V = (A, B) \in \mathcal{F}$. Because $A \in S$ (or $A \leq a \leq t_{k-1}$), and there are no elements of S inside (t_{k-1}, t_k), we must have $A \leq t_{k-1}$. Similarly, $B \geq t_k$. Therefore, $(t_{k-1}, t_k) \subset V$.

Step 6: Splitting the Darboux Sum.
Let $M_k = \sup_{[t_{k-1}, t_k]} f$ and $m_k = \inf_{[t_{k-1}, t_k]} f$. We partition the index set $\{1, \ldots, N\}$ into two disjoint classes to avoid double-counting:

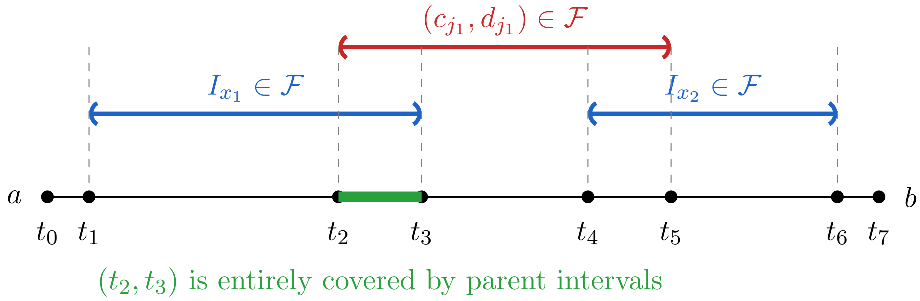

Figure 2: The endpoints of the finite subcover directly project downwards to form the partition grid t_k. Because the grid is constructed exactly from these endpoints, any gap between consecutive partition points (t_{k-1}, t_k) is cleanly swallowed by the parent covers without overlap.

- **The Good Set ($\mathcal{G}$):** Indices k where (t_{k-1}, t_k) is contained in I_{x_i} for some $i \in \{1, \ldots, n\}$.
- **The Bad Set ($\mathcal{B}$):** All remaining indices. Since they are not in $\mathcal{G}$, these subintervals *must* be contained in a quarantine interval (c_{j_s}, d_{j_s}) for some s.

Bounding $\mathcal{G}$: If $k \in \mathcal{G}$, taking closures yields $[t_{k-1}, t_k] \subset \overline{I_{x_i}}$. By Step 3, the oscillation of f is bounded: $M_k - m_k \leq \frac{\varepsilon}{2(b-a)}$. Thus:

$$\sum_{k \in \mathcal{G}} (M_k - m_k)\Delta t_k \leq \frac{\varepsilon}{2(b-a)} \sum_{k \in \mathcal{G}} \Delta t_k \leq \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

Bounding $\mathcal{B}$: If $k \in \mathcal{B}$, the global bound $|f| \leq M$ implies $M_k - m_k \leq 2M$. Because the open subintervals (t_{k-1}, t_k) for $k \in \mathcal{B}$ are mutually disjoint, and their union is contained within $\bigcup_{j=1}^{\infty}(c_j, d_j)$, the sum of their lengths cannot exceed the length of the covering intervals:

$$\sum_{k \in \mathcal{B}} \Delta t_k \leq \sum_{j=1}^{\infty}(d_j - c_j) < \frac{\varepsilon}{4M}.$$

$$\sum_{k \in \mathcal{B}} (M_k - m_k)\Delta t_k \leq 2M \sum_{k \in \mathcal{B}} \Delta t_k < 2M \left(\frac{\varepsilon}{4M}\right) = \frac{\varepsilon}{2}.$$

Conclusion:
Adding the split sums together yields the required bound on the Darboux sum difference:

$$U(f, P) - L(f, P) = \sum_{k \in \mathcal{G}} (M_k - m_k)\Delta t_k + \sum_{k \in \mathcal{B}} (M_k - m_k)\Delta t_k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is Riemann integrable. □

2 Discussion: Strategies, Robustness, and Applications

2.1 Architecture: The ε-Budgeting Bifurcation

Riemann integration essentially computes bounded area via the relation Area $\approx$ Height $\times$ Width. The brilliance of this proof lies in mathematically formalizing an orthogonal "Two-Regime" strategy that successfully decouples height from width:

- **The Measure Theoretic Regime (The Bad Set):** We have no control over the Height (bounded only by $2M$), so we forcefully restrict the **Width** using the Lebesgue measure zero property ($< \frac{\varepsilon}{4M}$).
- **The Topological Regime (The Good Set):** We have no control over the Width (could be as wide as $b - a$), so we forcefully restrict the **Height** via uniform bounds of local oscillation ($< \frac{\varepsilon}{2(b-a)}$).

The disjoint index partition trick cleanly splits the sum to handle both regimes without duplicating or crossing index bounds.

2.2 Robustness and Pathological Densities

Because this proof completely abstracts away the physical layout of the discontinuity set E, it is remarkably robust. Standard elementary calculus proofs manually construct partitions around a finite number of discontinuities, but this proof trivially digests severe pathologies.

It instantly guarantees that highly pathological functions—like **Thomae's "Popcorn" function**, which is discontinuous at *every single rational number*—are easily Riemann integrable, since countable sets intrinsically have Lebesgue measure zero. It even holds for uncountably infinite discontinuity sets, provided their measure is zero (e.g., continuous everywhere except on a Cantor Set).

2.3 Applications: The Composition of Functions

A notoriously difficult theorem to prove from first principles (using purely Darboux partitions) is that the composition of well-behaved functions is integrable. Using Lebesgue's Criterion, it becomes an elegant, one-line corollary.

Corollary 2. *If $f : [a, b] \to [c, d]$ is Riemann integrable, and $g : [c, d] \to \mathbb{R}$ is continuous, then the composition $g \circ f$ is Riemann integrable on $[a, b]$.*

Proof. Because g is continuous everywhere on its domain, a discontinuity in the composition $(g \circ f)(x)$ can only occur if f itself is discontinuous at x. Therefore, the discontinuity set obeys $D_{g \circ f} \subseteq D_f$. Since f is Riemann integrable, D_f has Lebesgue measure zero. Subsets of measure-zero sets automatically have measure zero, meaning $D_{g \circ f}$ has measure zero. Since continuous functions are bounded on compact domains, $g \circ f$ is bounded. By Lebesgue's Criterion, $g \circ f$ is Riemann integrable. $\square$