A Weighting Argument for Pointwise Convergence to Convergence in Measure

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Abstract

Let (f_n) be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$ such that $f_n(x) \to 0$ for every $x \in \mathbb{R}$. We give a detailed proof that there exists a strictly positive measurable function h such that $f_n h \to 0$ in measure. The main proof is an expanded version of a Math StackExchange answer of the author: it uses a finite-measure exhaustion, Egorov's theorem on each finite-measure piece, and a pointwise-boundedness stratification. The construction actually proves the stronger conclusion that $f_n h \to 0$ uniformly on a conull measurable subset of $\mathbb{R}$. We also include a shorter dominated-convergence proof of the original statement and record the corresponding sigma-finite version.

1 Introduction

On a finite measure space, almost everywhere convergence implies convergence in measure. On an infinite measure space, such as $(\mathbb{R}, m)$ with Lebesgue measure, this implication is false in general. The problem considered here asks whether one can multiply by a positive measurable weight in order to force convergence in measure.

The answer is yes. More precisely, if $f_n : \mathbb{R} \to \mathbb{R}$ are measurable and $f_n(x) \to 0$ pointwise, then there is a strictly positive measurable function $h : \mathbb{R} \to (0, \infty)$ such that

$$f_n h \longrightarrow 0$$

in measure. The construction below proves something stronger: h can be chosen so that $f_n h \to 0$ uniformly on a measurable set whose complement has Lebesgue measure zero.

The key idea is to combine two elementary consequences of pointwise convergence. First, after decomposing $\mathbb{R}$ into countably many finite-measure sets, Egorov's theorem gives countably many large pieces on which the convergence is uniform. Second, at each fixed point x, the scalar sequence $(f_n(x))_n$ is bounded, so $\mathbb{R}$ is the union of the level sets on which the pointwise envelope $\sup_n |f_n|$ is bounded. The weight h is then made small on the intersections of these two decompositions.

Throughout this note, convergence in measure on $\mathbb{R}$ means the global condition

$$m(\{x \in \mathbb{R} : |g_n(x)| > \varepsilon\}) \longrightarrow 0 \qquad \text{for every } \varepsilon > 0.$$

Equivalently, one may use $\geq \varepsilon$ instead of $> \varepsilon$.

2 The strengthened result on $\mathbb{R}$

We first state the stronger form proved by the weighting construction.

Theorem 2.1 (Conull uniformization by a positive weight)**.** *Let $f_n : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions such that*

$$f_n(x) \to 0 \qquad \text{for every } x \in \mathbb{R}.$$

Then there exist a measurable set $A \subseteq \mathbb{R}$ and a strictly positive finite-valued measurable function $h : \mathbb{R} \to (0, \infty)$ such that

$$m(\mathbb{R} \setminus A) = 0$$

and

$$\sup_{x \in A} |f_n(x)h(x)| \longrightarrow 0.$$

In particular, $f_n h \to 0$ in measure on $\mathbb{R}$.

The proof is split into two organizing lemmas.

Lemma 2.2 (Countably many uniform-convergence pieces)**.** *There exist measurable sets $A_1, A_2, \dots \subseteq \mathbb{R}$ such that*

$$m\left(\mathbb{R} \setminus \bigcup_{i=1}^{\infty} A_i\right) = 0,$$

and such that $f_n \to 0$ uniformly on each A_i.

Proof. Choose a finite-measure exhaustion of $\mathbb{R}$; for example, take

$$E_r = [-r, r], \qquad r \geq 1.$$

For each pair $r, k \geq 1$, Egorov's theorem applied on E_r gives a measurable set $A_{r,k} \subseteq E_r$ such that

$$m(E_r \setminus A_{r,k}) < \frac{1}{k}$$

and $f_n \to 0$ uniformly on $A_{r,k}$.

For fixed r, we have

$$E_r \setminus \bigcup_{k=1}^{\infty} A_{r,k} = \bigcap_{k=1}^{\infty} (E_r \setminus A_{r,k}),$$

and hence, for every k,

$$m\left(E_r \setminus \bigcup_{k=1}^{\infty} A_{r,k}\right) \leq m(E_r \setminus A_{r,k}) < \frac{1}{k}.$$

Therefore

$$m\left(E_r \setminus \bigcup_{k=1}^{\infty} A_{r,k}\right) = 0.$$

Let

$$A := \bigcup_{r,k \geq 1} A_{r,k}.$$

Since $\mathbb{R} = \bigcup_r E_r$, it follows that

$$\mathbb{R} \setminus A \subseteq \bigcup_{r=1}^{\infty} \left(E_r \setminus \bigcup_{k=1}^{\infty} A_{r,k} \right),$$

so $m(\mathbb{R} \setminus A) = 0$. Finally, reindex the countable family $\{A_{r,k} : r, k \geq 1\}$ as $A_1, A_2, \ldots$. Each reindexed set is measurable, and $f_n \to 0$ uniformly on each one. □

Lemma 2.3 (Pointwise boundedness stratification)**.** *Define*

$$M(x) := \sup_{n \geq 1} |f_n(x)|.$$

Then M is measurable and finite everywhere. Consequently, the sets

$$B_j := \{x \in \mathbb{R} : M(x) \leq j\}, \qquad j \geq 1,$$

are measurable and satisfy

$$\mathbb{R} = \bigcup_{j=1}^{\infty} B_j.$$

Moreover, on B_j one has

$$|f_n(x)| \leq j \qquad \text{for every } n \geq 1.$$

Proof. Since each f_n is measurable, so is $|f_n|$, and therefore the countable supremum

$$M = \sup_{n \geq 1} |f_n|$$

is measurable. For each fixed $x \in \mathbb{R}$, the numerical sequence $(f_n(x))_{n \geq 1}$ converges to 0, hence is bounded. Thus $M(x) < \infty$ for every x. The remaining assertions follow immediately from the definition of B_j. □

Proof of Theorem 2.1. Let $A = \bigcup_i A_i$, where the sets A_i come from Lemma 2.2. Let B_j be the measurable sets from Lemma 2.3. Define

$$h(x) := \sum_{i,j \geq 1} \frac{1}{j 2^{ij}} \mathbf{1}_{A_i \cap B_j}(x) + \mathbf{1}_{\mathbb{R} \setminus A}(x).$$

This is a nonnegative measurable function, being a countable sum of nonnegative measurable functions.

We first check that h is finite-valued and strictly positive. Since

$$\sum_{i,j \geq 1} \frac{1}{j 2^{ij}} \leq \sum_{i,j \geq 1} \frac{1}{2^{ij}} = \sum_{i=1}^{\infty} \frac{1}{2^i - 1} \leq \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = 2,$$

the defining series is bounded above by 2 everywhere. Hence h is finite. If $x \in \mathbb{R} \setminus A$, then $h(x) \geq 1$. If $x \in A$, then $x \in A_i$ for some i, and since $\mathbb{R} = \bigcup_j B_j$, also $x \in B_j$ for some j. The corresponding summand is then positive. Thus $h(x) > 0$ for every $x \in \mathbb{R}$.

It remains to prove uniform convergence on A. Fix $\varepsilon > 0$. Because

$$\sum_{i,j \geq 1} 2^{-ij} < \infty,$$

we may choose $s \geq 1$ so large that

$$\sum_{\max\{i,j\}>s} 2^{-ij} < \frac{\varepsilon}{2}.$$

For $x \in A$, the term $\mathbf{1}_{\mathbb{R}\setminus A}(x)$ vanishes, and therefore

$$\begin{aligned} |f_n(x)h(x)| &\leq \sum_{i,j\geq 1} \frac{|f_n(x)|}{j2^{ij}} \mathbf{1}_{A_i \cap B_j}(x) \\ &= \sum_{1\leq i,j\leq s} \frac{|f_n(x)|}{j2^{ij}} \mathbf{1}_{A_i \cap B_j}(x) + \sum_{\max\{i,j\}>s} \frac{|f_n(x)|}{j2^{ij}} \mathbf{1}_{A_i \cap B_j}(x). \end{aligned}$$

On $A_i \cap B_j$, Lemma 2.3 gives $|f_n(x)| \leq j$. Hence the tail is bounded uniformly in $x \in A$ and n by

$$\sum_{\max\{i,j\}>s} 2^{-ij} < \frac{\varepsilon}{2}.$$

For the finite block, we have

$$\sup_{x\in A} \sum_{1\leq i,j\leq s} \frac{|f_n(x)|}{j2^{ij}} \mathbf{1}_{A_i \cap B_j}(x) \leq \sum_{1\leq i,j\leq s} \frac{1}{j2^{ij}} \sup_{x\in A_i} |f_n(x)|.$$

For each fixed i, $f_n \to 0$ uniformly on A_i, and the sum above has only finitely many indices. Therefore the finite block tends to 0 uniformly on A. Thus there exists N such that for all $n \geq N$,

$$\sup_{x\in A} \sum_{1\leq i,j\leq s} \frac{|f_n(x)|}{j2^{ij}} \mathbf{1}_{A_i \cap B_j}(x) < \frac{\varepsilon}{2}.$$

Combining the finite-block estimate with the tail estimate gives

$$|f_n(x)h(x)| < \varepsilon \qquad \text{for all } x \in A \text{ and all } n \geq N.$$

Hence

$$\sup_{x\in A} |f_n(x)h(x)| \to 0.$$

Finally, since $m(\mathbb{R} \setminus A) = 0$, uniform convergence on A implies convergence in measure. Indeed, given $\varepsilon > 0$, for all sufficiently large n,

$$\{x \in \mathbb{R} : |f_n(x)h(x)| > \varepsilon\} \subseteq \mathbb{R} \setminus A,$$

and the right-hand side has measure zero. Therefore $f_n h \to 0$ in measure. □

3 A bookkeeping-light version of the same construction

The double series in the preceding proof is one natural way to combine the Egorov pieces A_i and the boundedness pieces B_j. A slightly cleaner presentation is obtained by first making the intersections disjoint.

Enumerate all pairs $(i,j) \in \mathbb{N}^2$ as

$$(i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots,$$

and set

$$C_p := A_{i_p} \cap B_{j_p}.$$

Now disjointify these sets by defining

$$D_1 = C_1, \qquad D_p = C_p \setminus \bigcup_{q<p} C_q \quad (p \geq 2).$$

Then the D_p are measurable, pairwise disjoint, and

$$A = \bigcup_{p=1}^{\infty} D_p.$$

Moreover, if $x \in D_p$, then $x \in A_{i_p}$ and $x \in B_{j_p}$, so $f_n \to 0$ uniformly on D_p and

$$|f_n(x)| \leq j_p \qquad (x \in D_p,\ n \geq 1).$$

Define instead

$$\widetilde{h}(x) := \sum_{p=1}^{\infty} \frac{1}{p j_p} \mathbf{1}_{D_p}(x) + \mathbf{1}_{\mathbb{R} \setminus A}(x).$$

Because the D_p are disjoint, this simply says that

$$\widetilde{h}(x) = \frac{1}{p j_p} \quad \text{if } x \in D_p, \qquad \widetilde{h}(x) = 1 \quad \text{if } x \notin A.$$

Thus $\widetilde{h}$ is measurable, finite-valued, and strictly positive.

The proof of uniform convergence is now particularly transparent. Fix $\varepsilon > 0$ and choose P so large that $1/P < \varepsilon$. If $p > P$ and $x \in D_p$, then for every n,

$$|f_n(x)\widetilde{h}(x)| \leq j_p \cdot \frac{1}{p j_p} = \frac{1}{p} < \varepsilon.$$

On the finite union $D_1 \cup \cdots \cup D_P$, uniform convergence follows from uniform convergence on the corresponding finitely many sets A_{i_p}. More explicitly, for each $p \leq P$, choose N_p so that

$$\sup_{x \in D_p} |f_n(x)| < \varepsilon p j_p \qquad (n \geq N_p).$$

With $N = \max_{1 \leq p \leq P} N_p$, we get for every $n \geq N$ and every $x \in D_p$, $p \leq P$,

$$|f_n(x)\widetilde{h}(x)| < \varepsilon.$$

Together with the tail estimate for $p > P$, this proves

$$\sup_{x \in A} |f_n(x)\widetilde{h}(x)| \leq \varepsilon$$

for all sufficiently large n. Hence $f_n \widetilde{h} \to 0$ uniformly on A, and therefore in measure.

Remark 3.1. This disjointified proof is the same idea as the double-series proof. The sets A_i encode where convergence is already uniform; the sets B_j encode where the whole sequence is uniformly bounded; and the coefficients of the weight shrink sufficiently fast as one moves through the countable list of pieces.

4 A shorter proof of the original convergence-in-measure statement

The stronger conull-uniform result is useful conceptually, but the original question also has a concise dominated-convergence proof.

Let

$$M(x) := \sup_{n\geq 1} |f_n(x)|.$$

As above, M is measurable and finite everywhere. Set

$$g(x) := e^{-|x|}$$

and define

$$h_0(x) := \frac{g(x)}{1+M(x)}.$$

Then h_0 is measurable and strictly positive. Moreover,

$$|f_n(x)h_0(x)| = |f_n(x)|\frac{g(x)}{1+M(x)} \leq g(x).$$

Since $g \in L^1(\mathbb{R})$ and $f_n(x)h_0(x) \to 0$ pointwise, the dominated convergence theorem gives

$$\int_{\mathbb{R}} |f_n h_0| \, dm \longrightarrow 0.$$

Thus $f_n h_0 \to 0$ in $L^1(\mathbb{R})$, and hence in measure, since Markov's inequality gives

$$m\big(\{x \in \mathbb{R} : |f_n(x)h_0(x)| > \varepsilon\}\big) \leq \frac{1}{\varepsilon}\int_{\mathbb{R}} |f_n h_0| \, dm \longrightarrow 0$$

for every $\varepsilon > 0$.

Remark 4.1. The dominated-convergence proof is shorter, but it proves a different kind of strengthening: L^1 convergence rather than uniform convergence on a conull set. The Egorov-based construction is more structural. It shows that after multiplication by a positive measurable weight, the convergence can be made uniform away from a null set, even though the underlying measure space is infinite.

5 The sigma-finite form

Nothing in the Egorov argument is special to the topology of $\mathbb{R}$. The relevant measure-theoretic hypothesis is sigma-finiteness.

Theorem 5.1 (Sigma-finite version)**.** *Let $(X, \mathcal{A}, \mu)$ be a sigma-finite measure space, and let $f_n : X \to \mathbb{R}$ be measurable functions such that $f_n(x) \to 0$ for μ-almost every $x \in X$. Then there exist a measurable set $A \subseteq X$ with $\mu(X \setminus A) = 0$ and a strictly positive finite-valued measurable function $h : X \to (0, \infty)$ such that*

$$\sup_{x\in A} |f_n(x)h(x)| \longrightarrow 0.$$

Consequently $f_n h \to 0$ in measure.

Proof. Let

$$X_0 := \{x \in X : f_n(x) \to 0\}.$$

This set is measurable, since

$$X_0 = \bigcap_{\ell=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{x \in X : |f_n(x)| < 1/\ell\},$$

and it is conull by hypothesis. Choose a countable cover $X = \bigcup_{r=1}^{\infty} E_r$ with $\mu(E_r) < \infty$. Applying Egorov's theorem on each finite-measure set $E_r \cap X_0$, exactly as in Lemma 2.2, gives countably many measurable sets $A_1, A_2, \cdots \subseteq X_0$ such that

$$\mu\left(X \setminus \bigcup_{i=1}^{\infty} A_i\right) = 0,$$

and $f_n \to 0$ uniformly on each A_i. Put

$$A := \bigcup_{i=1}^{\infty} A_i.$$

On X_0, define the envelope

$$M(x) := \sup_{n \geq 1} |f_n(x)|.$$

It is finite on X_0. For $j \geq 1$, define

$$B_j := X_0 \cap \{x \in X : M(x) \leq j\}.$$

Then the B_j are measurable, $X_0 = \bigcup_j B_j$, and $|f_n| \leq j$ on B_j for all n.

Now define

$$h(x) := \sum_{i,j \geq 1} \frac{1}{j 2^{ij}} \mathbf{1}_{A_i \cap B_j}(x) + \mathbf{1}_{X \setminus A}(x).$$

The same estimates as in the proof of Theorem 2.1 show that h is measurable, finite-valued, strictly positive everywhere, and that $f_n h \to 0$ uniformly on A. Since $\mu(X \setminus A) = 0$, convergence in measure follows. □

The shorter L^1 proof also has a sigma-finite version.

Proposition 5.2 (Sigma-finite dominated-convergence proof)**.** *Let $(X, \mathcal{A}, \mu)$ be sigma-finite, and suppose $f_n : X \to \mathbb{R}$ are measurable and $f_n \to 0$ almost everywhere. Then there is a strictly positive finite-valued measurable function h such that*

$$\int_X |f_n h| \, d\mu \longrightarrow 0.$$

In particular, $f_n h \to 0$ in measure.

Proof. Let $X_0 = \{x : f_n(x) \to 0\}$, and let

$$M(x) := \sup_{n \geq 1} |f_n(x)|.$$

The function M is an extended-valued measurable function and is finite on X_0. Define the finite-valued measurable function

$$\widetilde{M}(x) := \begin{cases} M(x), & x \in X_0, \\ 0, & x \notin X_0. \end{cases}$$

Since μ is sigma-finite, choose measurable sets E_r with $X = \bigcup_r E_r$ and $\mu(E_r) < \infty$. Passing to the disjoint refinement

$$F_1 = E_1, \qquad F_r = E_r \setminus \bigcup_{q<r} E_q \quad (r \geq 2),$$

we obtain a measurable partition of X with $\mu(F_r) < \infty$. Define

$$g(x) := \sum_{r=1}^{\infty} \frac{2^{-r}}{1 + \mu(F_r)} \mathbf{1}_{F_r}(x).$$

Then g is strictly positive, finite-valued, measurable, and integrable, since

$$\int_X g \, d\mu = \sum_{r=1}^{\infty} \frac{2^{-r}\mu(F_r)}{1 + \mu(F_r)} \leq \sum_{r=1}^{\infty} 2^{-r} < \infty.$$

Define

$$h(x) := \frac{g(x)}{1 + \widetilde{M}(x)}.$$

Then h is strictly positive, finite-valued, and measurable. On the conull set X_0,

$$|f_n h| \leq g \qquad \text{and} \qquad f_n h \to 0.$$

The dominated convergence theorem therefore gives

$$\int_X |f_n h| \, d\mu \to 0.$$

The convergence in measure follows from Markov's inequality. □

6 Concluding remarks

The original construction is a useful pattern. When pointwise convergence is too weak globally, one can often separate the problem into two countable stratifications:

(i) pieces on which convergence is uniform, supplied here by Egorov's theorem on finite-measure pieces;

(ii) pieces on which the whole sequence is uniformly bounded, supplied here by the pointwise envelope $M = \sup_n |f_n|$.

The weight is then chosen small enough on the intersections of the two stratifications. The role of sigma-finiteness is precisely to make the first stratification countable through finite-measure exhaustion. The role of pointwise convergence is not only to give the limiting value 0, but also to guarantee pointwise boundedness of the sequence, which is what makes the second stratification possible.

References

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