A Note on Disjoint Decompositions of $[0, 1)$ into Closed Sets

Henry Shin

October 10, 2024

Theorem 1. *There does not exist a sequence $(E_n)_{n\geq 1}$ of pairwise disjoint closed subsets of $\mathbb{R}$ such that*

$$[0, 1) = \bigcup_{n=1}^{\infty} E_n.$$

Proof. Assume, for contradiction, that such a sequence (E_n) exists.

Step 1: Pass to the compact interval $[0, 1]$. Let

$$X = [0, 1], \qquad F_0 = \{1\}, \qquad F_n = E_n \ \ (n \geq 1).$$

Then $(F_n)_{n\geq 0}$ is a sequence of pairwise disjoint closed subsets of X whose union is all of X:

$$X = \bigcup_{n=0}^{\infty} F_n.$$

For each $n \geq 0$, let

$$I_n = \operatorname{int}_X(F_n)$$

denote the interior of F_n relative to X, and define

$$B_n = F_n \setminus I_n.$$

Thus B_n is the part of F_n left over after removing its relative interior.

Now set

$$U = \bigcup_{n=0}^{\infty} I_n, \qquad P = X \setminus U.$$

Since U is open in X, the set P is closed in X. As X is compact, P is compact and therefore a complete metric space.

Because the sets F_n are pairwise disjoint and cover X, we have

$$P = \bigcup_{n=0}^{\infty} B_n,$$

and each B_n is closed in P.

Step 2: Apply the Baire Category Theorem. Since P is a complete metric space and is expressed as a countable union of closed subsets, the Baire Category Theorem implies that at least one of the sets B_k has nonempty interior relative to P. Thus there exist an index $k \geq 0$ and a nonempty set $O \subseteq X$, open in X, such that

$$\varnothing \neq O \cap P \subseteq B_k.$$

Choose a point $t \in O \cap P$. Since O is open in the subspace $X = [0, 1]$, there exists a nondegenerate interval $J \subseteq X$, open in X, such that

$$t \in J \subseteq O.$$

Then

$$J \cap P \subseteq O \cap P \subseteq B_k.$$

Step 3: Use disjointness and connectedness. Fix any $m \neq k$. Since

$$F_m = I_m \cup B_m$$

and $B_m \subseteq P$, the inclusion $J \cap P \subseteq B_k$ implies that

$$J \cap B_m = \varnothing.$$

Therefore every point of J lies either in $X \setminus F_m$ or in I_m, so

$$J \subseteq (X \setminus F_m) \cup I_m.$$

The sets $X \setminus F_m$ and I_m are disjoint open subsets of X. Since J is connected, it follows that either

$$J \subseteq X \setminus F_m \qquad \text{or} \qquad J \subseteq I_m.$$

The second alternative is impossible, because $t \in B_k \subseteq F_k$ while $F_k \cap F_m = \varnothing$, so $t \notin F_m$, hence $t \notin I_m$. Thus

$$J \subseteq X \setminus F_m \qquad \text{for every } m \neq k.$$

Since the sets F_n partition X, we conclude that

$$J \subseteq F_k.$$

But J is open in X, so this implies

$$J \subseteq \operatorname{int}_X(F_k) = I_k.$$

In particular,

$$t \in I_k.$$

On the other hand, $t \in J \cap P \subseteq B_k = F_k \setminus I_k$, so $t \notin I_k$. This is a contradiction.

Hence no such decomposition of $[0, 1)$ exists. □

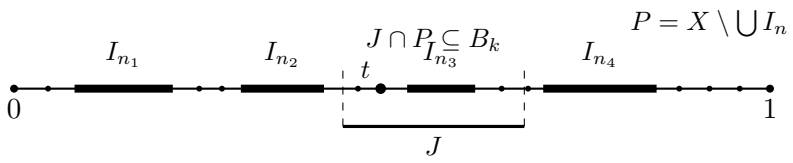

Figure 1: Schematic picture of the proof. The thick subintervals represent relative interiors I_n, while the remaining closed set P consists of the points not lying in any I_n. Baire yields an interval J such that $J \cap P$ lies entirely in a single boundary piece B_k. Connectedness then forces all of J into F_k, contradicting the fact that $t \in B_k \subseteq F_k \setminus I_k$.